

同步练习 P23-24 理学院 廖国勇 2016.9.23

一. 填空题

$$1. \lim_{x \rightarrow \infty} \frac{\sin 2x}{x} = \lim_{x \rightarrow \infty} \boxed{(\sin 2x)} \cdot \boxed{\frac{1}{x}} = 0$$

↑ 有界量 ↑ 无穷小

$$2. \lim_{x \rightarrow 0} \frac{x^2 \sin \frac{1}{x}}{\tan x} = \lim_{x \rightarrow 0} \frac{x}{\tan x} \cdot x \cdot \sin \frac{1}{x} = \lim_{x \rightarrow 0} x \cdot \sin \frac{1}{x} = 0$$

↑ 无穷小 ↑ 有界量

$$3. \lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x^2-1} - \frac{\sin x}{x} \right) = \lim_{x \rightarrow \infty} \frac{x^2+1}{x^2-1} - \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 1 - 0 = 1$$

4. 设当 $x \rightarrow 1$ 时, $\sqrt{x}-1$ 与 $k(x-1)$ 等价, 则 $k = \underline{\hspace{2cm}}$

解: $1 = \lim_{x \rightarrow 1} \frac{k(x-1)}{\sqrt{x}-1} = \lim_{x \rightarrow 1} k(\sqrt{x}+1) = 2k \Rightarrow k = \frac{1}{2}$

二. 计算题

$$1. \lim_{x \rightarrow \infty} \frac{x - \sin x}{x + \sin x} = \lim_{x \rightarrow \infty} \frac{1 - \frac{\sin x}{x}}{1 + \frac{\sin x}{x}} = \frac{1-0}{1+0} = 1$$

$$2. \lim_{n \rightarrow \infty} \frac{n(\sqrt{1+\sin^2 \frac{1}{n}} - 1)}{\arcsin \frac{2}{n}} = \lim_{n \rightarrow \infty} \frac{n \cdot \frac{1}{2} \sin^2 \frac{1}{n}}{\frac{2}{n}} = \lim_{n \rightarrow \infty} \frac{n \cdot \frac{1}{2} \cdot (\frac{1}{n})^2}{\frac{2}{n}} = \frac{1}{4}$$

(答案 $\frac{1}{4}$ 系 $\frac{1}{4}$)

3. 利用等价无穷小替代求极限

$$(1) \lim_{x \rightarrow 0} \frac{1 - \sqrt{\cos x}}{x(1 - \cos \sqrt{x})} = \lim_{x \rightarrow 0} \frac{-(\sqrt{1 + (\cos x - 1)} - 1)}{x \cdot \frac{1}{2} (\sqrt{x})^2}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{1}{2}(\cos x - 1)}{\frac{1}{2}x^2} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}$$

$$(2) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{(\sqrt[3]{1+x^2} - 1)(\sqrt{1+\sin x} - 1)} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^3}{\frac{1}{3}x^2 \cdot \frac{1}{2}\sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x}{\frac{1}{3} \cdot \frac{1}{2} \sin x} = 3$$

$$4. \lim_{x \rightarrow \infty} \left(\frac{3x^2+1}{4x+3} \sin \frac{5}{x} \right) = \lim_{x \rightarrow \infty} \left(\frac{3x^2+1}{4x+3} \cdot \frac{5}{x} \right) = \frac{3x}{4} = \frac{15}{4}$$

P25 5 设当 $x \rightarrow \infty$ 时, 函数 $f(x) = \frac{x^2-2x}{x+1} - ax+b$ 为无穷小量, 求 a, b

$$\text{解: } \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2-2x}{x+1} - ax+b = \lim_{x \rightarrow \infty} \frac{(1-a)x^2 + (b-a-2)x + b}{x+1} = 0$$

$$\Rightarrow \begin{cases} 1-a=0 \\ b-a-2=0 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=3 \end{cases}$$

三. 证明题

1. 当 $x \rightarrow 0$ 时, $\sec x - 1 \sim \frac{x^2}{2}$

$$\text{证明: } \lim_{x \rightarrow 0} \frac{\sec x - 1}{\frac{x^2}{2}} = \lim_{x \rightarrow 0} \frac{\frac{1-\cos x}{\cos x}}{\frac{x^2}{2}} = \lim_{x \rightarrow 0} \frac{\frac{\frac{1}{2}x^2}{\cos x}}{\frac{1}{2}x^2} = 1$$

2. 设当 $x \rightarrow \infty$ 时 $f(x)$ 为无穷大, 极限 $\lim_{x \rightarrow \infty} g(x)$ 存在, 证明: 当 $x \rightarrow \infty$ 时, $f(x) + g(x)$ 为无穷大

证明: 设 $\lim_{x \rightarrow \infty} g(x) = A$, 于是 $\exists \delta_1 > 0$, 当 $|x| > \delta_1$ 时, $|g(x) - A| < 1$

即得 $A-1 < g(x) < A+1$, 当然 $|g(x)| < |A|+1$
 $\forall M > 0$,

$\lim_{x \rightarrow \infty} f(x) = \infty \Rightarrow \exists \delta_2 > 0$, 当 $|x| > \delta_2$ 时, $|f(x)| > M + |A| + 1$

取 $\delta = \max\{\delta_1, \delta_2\}$, 当 $|x| > \delta$ 时, $|f(x)| > M + |A| + 1$ 与 $|g(x)| < |A| + 1$

同时成立, 从而 $|f(x) + g(x)| \geq |f(x)| - |g(x)| > M + |A| + 1 - |A| - 1 = M$

$\therefore f(x) + g(x)$ 为无穷大.